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一类不确定分数阶广义系统的基于观测器鲁棒预测控制

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摘要:本文针对一类不确定分数阶广义系统,研究其基于观测器的鲁棒预测控制问题。首先,基于分数阶微积分性质构造带有合适误差项的 Lyapunov 函数,运用 Lyapunov 稳定性理论求解其优化问题;其次,通过应用线性矩阵不等式与锥补线性算法,推导出鲁棒预测控制器存在的充分条件,并证明该条件满足闭环系统的可容许性;最后,借助仿真实验验证了控制策略的有效性。

关键词:不确定分数阶广义系统;观测器;鲁棒预测控制;Lyapunov 函数

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分数阶系统是一类较经典整数阶系统更加复杂的系统,该系统能更好地描述研究对象的物理特性,经常被应用在粘弹性系统、电化学、经济和生物系统等领域[1—3]。分数阶微积分的典型应用是对新型冠状病毒(COVID-19)[4]进行建模,由于其在理论研究和实际应用中的重要性越来越受到人们的关注,尤其体现在稳定性分析[5—6]、控制器综合[7—8]等方面。目前,在分数阶广义系统的控制方面已经做了很多工作,其中包括最优控制、滑模控制等[9—10]。对于分数阶广义系统容许性的研究,文献[11]利用线性矩阵不等式得到了系统可容许的充要条件;文献[12]针对阶数满足 $0<\alpha<2$ 的分数阶广义系统,利用线性矩阵不等式和广义线性矩阵不等式分别给出了保证系统稳定性和容许性的充要条件。

在实际控制系统中,由于建模误差、环境变换等诸多因素的干扰,系统内部不可避免地存在各种不确定性,这些不确定性因素可能会对系统的性能产生负面影响,导致系统性能退化[13]。鲁棒预测控制将鲁棒控制处理不确定性的能力与预测控制的滚动优化方法相结合,有效克服模型不确定性问题[14]。近年来,关于不确定分数阶系统的鲁棒预测控制问题,取得了诸多重要研究成果。文献[8]针对具有时变不确定性的广义系统,开展了鲁棒预测控制问题的深入研究;文献[13]探讨了在系统状态不能直接量测的情况下,针对具有范数有界不确定性的时滞广义系统,提出基于观测器的鲁棒预测控制算法;文献[14]进一步研究了具有不确定参数的分数阶系统的鲁棒预测控制相关问题。

在控制工程中,系统的状态通常不是完全已知的,为了解决这一问题,通常需要引入观测器估计系统的内部状态[15]。文献[16—20]主要研究了确定性分数阶系统的观测器设计问题。关于不确定分数阶系统的观测器设计问题,文献[21]利用线性矩阵不等式和奇异值分解,探讨了基于观测器的鲁棒控制方法,确立了不确定系统的稳定性条件。文献[22]进一步针对不确定分数阶广义系统,采用 Shur 补引理、线性矩阵不等式及锥补线性化算法,提出一种新的鲁棒预测控制方法。

基于以上研究成果,本文针对一类具有参数不确定性的分数阶广义系统,深入研究基于观测器的鲁棒预测控制问题与闭环系统容许性问题,并通过仿真算例验证了控制方法的可行性。

1 问题描述与预备知识

考虑一类含参数不确定性的分数阶广义系统

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$$\begin{cases}\boldsymbol{E}D^{\alpha}\boldsymbol{x}(t)=(\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t),\end{cases}\tag{1}$$

其中:$\alpha\in(0,1)$ 为分数阶系统的阶数,$D^{\alpha}\boldsymbol{x}(t)$ 表示Caputo定义下 $\boldsymbol{x}(t)$ 的 α 阶微分;$\boldsymbol{x}(t)\in\mathbf{R}^{n}$,$\boldsymbol{u}(t)\in\mathbf{R}^{m}$ 分别是系统的状态向量和控制输入,$\boldsymbol{y}(t)\in\mathbf{R}^{p}$ 是系统输出;$\boldsymbol{E}\in\mathbf{R}^{n\times n}$ 为奇异矩阵,且 $\operatorname{rank}(\boldsymbol{E})=r<n$;$\boldsymbol{A}$,$\boldsymbol{B}$,$\boldsymbol{C}$ 是具有适当维数的已知常数矩阵;$\Delta\boldsymbol{A}(t)$ 表示范数有界的参数不确定性时变矩阵,满足 $\Delta\boldsymbol{A}(t)=\boldsymbol{G}\boldsymbol{F}(t)\boldsymbol{H}$,$\boldsymbol{G}$,$\boldsymbol{H}$ 是具有适当维数的常数矩阵,$\boldsymbol{F}(t)$ 是未知且时变的矩阵函数,满足 $\boldsymbol{F}(t)\boldsymbol{F}^{\mathrm{T}}(t)\leqslant I$。

针对不确定分数阶广义系统(1),考虑以下性能指标:

$$\begin{cases}\min\limits_{\boldsymbol{u}(kT+i,kT)} J_k,\\ J_k=\max\limits_{\boldsymbol{F}(kT+i,kT),i\geqslant 0}\ {}_{\infty}I_0^{\alpha}\,\boldsymbol{x}^{\mathrm{T}}(kT+i,kT)\boldsymbol{Q}_1\boldsymbol{x}(kT+i,kT)+\boldsymbol{u}^{\mathrm{T}}(kT+i,kT)\boldsymbol{R}\boldsymbol{u}(kT+i,kT),\end{cases}\tag{2}$$

其中:$\boldsymbol{Q}_1>0$ 和 $\boldsymbol{R}>0$ 是对称加权正定矩阵;${}_{\infty}I_0^{\alpha}$ 表示初始化从0到 ∞ 的 α 阶积分。

对于系统(1),考虑如下形式的观测器:

$$\boldsymbol{E}D^{\alpha}\hat{\boldsymbol{x}}(t)=\boldsymbol{A}\hat{\boldsymbol{x}}(t)+\boldsymbol{B}\boldsymbol{u}(t)+\boldsymbol{L}(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)),\tag{3}$$

其中,$\hat{\boldsymbol{x}}(t)$ 表示状态预测值,$\boldsymbol{L}$ 为观测增益矩阵。设计观测器型的鲁棒预测控制器为

$$\boldsymbol{u}(kT+i,kT)=\boldsymbol{K}\hat{\boldsymbol{x}}(kT+i,kT),k\geqslant 0,i\geqslant 0,\tag{4}$$

其中:$\boldsymbol{K}\in\mathbf{R}^{n\times n}$ 是待定的控制增益矩阵;T 为采样周期,$\hat{\boldsymbol{x}}(kT+i,kT)$ 表示在 $kT+i$ 时刻的状态预测值,$\boldsymbol{u}(kT+i,kT)$ 表示在 $kT+i$ 时刻满足性能指标(2)的控制输入。

本文旨在设计鲁棒预测控制器(4),使得系统在每个采样时刻 kT 均能满足最优化问题(2),并在可行性条件满足时使得闭环系统是容许的。

引理1[22] 若 $\alpha\in(0,1)$,$\boldsymbol{x}(t)=(\boldsymbol{x}_1(t),\boldsymbol{x}_2(t),\cdots,\boldsymbol{x}_n(t))^{\mathrm{T}}\in\mathbf{R}^{n}$,$\boldsymbol{x}_i(t)\ (i=1,2,\cdots,n)$ 是连续可微函数,则对任意 $t>0$,存在正定矩阵 $\boldsymbol{P}\in\mathbf{R}^{n\times n}$,使得

$${}_0^cD^{\alpha}(\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{x}(t))\leqslant({}_0^cD^{\alpha}\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\,{}_0^cD^{\alpha}\boldsymbol{x}(t)。$$

引理2[23] 给定具有适当维数的矩阵 $\boldsymbol{S}_1$,$\boldsymbol{S}_2$,$\boldsymbol{S}_3$,且 $\boldsymbol{S}_1=\boldsymbol{S}_1^{\mathrm{T}}>0$,$\boldsymbol{S}_3=\boldsymbol{S}_3^{\mathrm{T}}>0$,则 $\boldsymbol{S}_1+\boldsymbol{S}_2\boldsymbol{S}_3^{-1}\boldsymbol{S}_2^{\mathrm{T}}<0$,当且仅当 $\begin{bmatrix}\boldsymbol{S}_1 & \boldsymbol{S}_2\\ \boldsymbol{S}_2^{\mathrm{T}} & \boldsymbol{S}_3\end{bmatrix}<0$。

引理3[23] 对任意具有适当维数的矩阵 $\boldsymbol{X}$,$\boldsymbol{Y}$,$\boldsymbol{Q}$,且 $\boldsymbol{Q}$ 是对称正定矩阵,则 $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{X}\leqslant\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{X}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{Y}$;特别地,当 $\boldsymbol{Q}=\varepsilon\boldsymbol{I}$ 时,$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}+\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{X}\leqslant\varepsilon\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}+\varepsilon^{-1}\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y}$。

引理4[24] 对任意矩阵 $\boldsymbol{\Pi}\in\mathbf{R}^{m\times n}(m<n)$,$\operatorname{rank}(\boldsymbol{\Pi})=m$,则 $\boldsymbol{\Pi}$ 存在奇异值分解 $\boldsymbol{\Pi}=U_1[\boldsymbol{S}_1\quad \boldsymbol{0}]\boldsymbol{V}_1^{\mathrm{T}}$,这里,$\boldsymbol{S}_1=\operatorname{diag}(\sigma_1,\sigma_2,\cdots,\sigma_m)\in\mathbf{R}^{m\times m}$,$\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_m>0$,$\boldsymbol{U}_1\in\mathbf{R}^{m\times m}$ 和 $\boldsymbol{V}_1\in\mathbf{R}^{n\times n}$ 是酉矩阵。

引理5[24] 给定矩阵 $\boldsymbol{\Pi}\in\mathbf{R}^{m\times n}(m<n)$,$\operatorname{rank}(\boldsymbol{\Pi})=m$,假设 $\boldsymbol{X}\in\mathbf{R}^{n\times n}$ 为对称矩阵,那么存在矩阵 $\overline{\boldsymbol{X}}\in\mathbf{R}^{m\times m}$ 满足 $\boldsymbol{\Pi}\boldsymbol{X}=\overline{\boldsymbol{X}}\boldsymbol{\Pi}$,当且仅当 $\boldsymbol{X}=\boldsymbol{V}\begin{bmatrix}\boldsymbol{X}_{11} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{X}_{22}\end{bmatrix}\boldsymbol{V}^{\mathrm{T}}$,$\boldsymbol{X}_{11}\in\mathbf{R}^{m\times m}$,$\boldsymbol{X}_{22}\in\mathbf{R}^{(n-m)\times(n-m)}$,$\boldsymbol{V}\in\mathbf{R}^{n\times n}$ 是酉矩阵。

2 基于观测器的鲁棒预测控制设计

为了有效处理由参数不确定性引起的优化问题,进而得到观测器型鲁棒预测控制器,首先构造带有合适误差项的Lyapunov函数,通过不等式假设条件将性能指标(2)转化为上界最小化问题。

根据Kronecker积的性质,则系统(1)等价为

$$(\boldsymbol{I}_2\otimes\boldsymbol{E})(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{x}(t))=(\boldsymbol{I}_2\otimes(\boldsymbol{A}+\Delta\boldsymbol{A}(t)))(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))+(\boldsymbol{I}_2\otimes\boldsymbol{B})(\boldsymbol{I}_2\otimes\boldsymbol{u}(t)),\tag{5}$$

则带有观测器的系统等价于

$$\begin{gathered}(\boldsymbol{I}_2\otimes\boldsymbol{E})(\boldsymbol{I}_2\otimes D^{\alpha}\hat{\boldsymbol{x}}(t))=\\ (\boldsymbol{I}_2\otimes\boldsymbol{A})(\boldsymbol{I}_2\otimes\hat{\boldsymbol{x}}(t))+(\boldsymbol{I}_2\otimes\boldsymbol{B})(\boldsymbol{I}_2\otimes\boldsymbol{u}(t))+(\boldsymbol{I}_2\otimes\boldsymbol{L})(\boldsymbol{I}_2\otimes(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t)))。\end{gathered}\tag{6}$$

考虑 Lyapunov 函数:

$$V(\boldsymbol{I}_2\otimes\boldsymbol{x}(t),\boldsymbol{I}_2\otimes\boldsymbol{e}(t))=(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}$$
$$(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))+(\boldsymbol{I}_2\otimes\boldsymbol{e}(t))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{T}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{e}(t)),\quad(7)$$

其中 $\begin{bmatrix}\boldsymbol{Q}_{11} & \boldsymbol{Q}_{12}\\ -\boldsymbol{Q}_{12} & \boldsymbol{Q}_{11}\end{bmatrix}>0,\begin{bmatrix}\boldsymbol{Q}_{21} & \boldsymbol{Q}_{22}\\ -\boldsymbol{Q}_{22} & \boldsymbol{Q}_{21}\end{bmatrix}>0$。

在每个采样时刻 kT, 假设

$$(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{x}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{T}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT))+$$
$$(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{x}(kT+i,kT))+$$
$$(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{e}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i,kT))+$$
$$(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{e}(kT+i,kT))\leqslant$$
$$-(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{Q}_1)(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT))-$$
$$(\boldsymbol{I}_2\otimes\boldsymbol{u}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{R})(\boldsymbol{I}_2\otimes\boldsymbol{u}(kT+i,kT)),\quad(8)$$

根据引理1,可得

$$D^{\alpha}V(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT),\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i,kT))\leqslant-(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{Q}_1)(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i,kT))-$$
$$(\boldsymbol{I}_2\otimes\boldsymbol{u}(kT+i,kT))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{R})(\boldsymbol{I}_2\otimes\boldsymbol{u}(kT+i,kT))。\quad(9)$$

将式(9)从 $i=0$ 到 $i=\infty$ 进行分数阶积分运算,得到 $J_k\leqslant V(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT),\boldsymbol{I}_2\otimes\boldsymbol{e}(kT))$。因此,将针对性能指标 J_k 求解最小化问题转化为对 $V(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT),\boldsymbol{I}_2\otimes\boldsymbol{e}(kT))$ 求最小化问题。

定理1 针对参数不确定分数阶广义系统(1),若存在状态反馈控制器(4)使 $V(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT),\boldsymbol{I}_2\otimes\boldsymbol{e}(kT))$ 最小,则观测器型预测控制器的增益矩阵为 $\boldsymbol{K}=(\boldsymbol{P}_0^{-1})^{\mathrm{T}}\boldsymbol{X}_1$,$\boldsymbol{P}_0$ 是对称正定矩阵, $\boldsymbol{P}_0,\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{Q},\gamma,\varepsilon_1,\varepsilon_2,\varepsilon_3$ 由以下优化问题求得

$$\min_{\gamma,\varepsilon_1,\varepsilon_2,\varepsilon_3,\boldsymbol{P}_0,\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{Q}}\boldsymbol{I}_2\otimes\gamma,$$

$$\begin{bmatrix}\boldsymbol{I}_2\otimes\gamma & \boldsymbol{I}_2\otimes(\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0) & \boldsymbol{I}_2\otimes(\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0)\\ * & \sum_{i=1}^{2}\boldsymbol{\Theta}_{i1}^{-1}\otimes\boldsymbol{P}_0 & \boldsymbol{0}\\ * & * & \sum_{i=1}^{2}\boldsymbol{\Theta}_{i1}^{-1}\otimes\boldsymbol{P}_0\end{bmatrix}\geqslant0,\quad(10)$$

$$\begin{bmatrix}\boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13}\\ * & \boldsymbol{\Phi}_{22} & \boldsymbol{0}\\ * & * & \boldsymbol{\Phi}_{33}\end{bmatrix}<0,\quad(11)$$

其中

$$\boldsymbol{\Phi}_{11}=\mathrm{sym}\left\{\sum_{i=1}^{2}\boldsymbol{\Theta}_{i1}\otimes\begin{bmatrix}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{A}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{X}_1 & -\boldsymbol{E}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{X}_1\\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{A}-\boldsymbol{E}^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{C}\end{bmatrix}\right\}+\mathrm{sym}\left\{\boldsymbol{I}_2\otimes\left(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}\end{bmatrix}\right)\right\}+$$
$$\sum_{i=1}^{3}\varepsilon_i\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\end{bmatrix}\right)+\begin{bmatrix}\boldsymbol{I}_2\otimes(\boldsymbol{Q}_1+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K}) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix},$$

$$\boldsymbol{\Phi}_{12}=\left[(\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))^{\mathrm{T}}\quad \boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{BX}_1 & -\boldsymbol{BX}_1\\ \boldsymbol{0} & -X_2\boldsymbol{C}\end{bmatrix}\right],$$

$$\boldsymbol{\Phi}_{13}=\left[\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G}\end{bmatrix}\quad \boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G}\end{bmatrix}\quad \boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\boldsymbol{G} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\boldsymbol{G}\end{bmatrix}\right],$$

$$\boldsymbol{\Phi}_{22}=-\boldsymbol{I}_4\otimes\begin{bmatrix}\boldsymbol{P}_0 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{P}_0\end{bmatrix},\boldsymbol{\Phi}_{33}=\begin{bmatrix}-\varepsilon_1 & 0 & 0\\ 0 & -\varepsilon_2 & 0\\ 0 & 0 & -\varepsilon_3\end{bmatrix}\otimes\boldsymbol{I}_{4n}。$$

证明 首先考虑 Lyapunov 函数(7),假设存在参数 γ 满足 $V(\boldsymbol{I}_2\otimes\boldsymbol{x}(t),\boldsymbol{I}_2\otimes\boldsymbol{e}(t))\leqslant\boldsymbol{I}_2\otimes\gamma$,则

$$(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))+$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{e}(t))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{e}(t))\leqslant\boldsymbol{I}_2\otimes\gamma。\quad(12)$$

设 $\boldsymbol{Q}_{11}=\boldsymbol{Q}_{21}=\boldsymbol{P}_0,\boldsymbol{Q}_{12}=\boldsymbol{Q}_{22}=0$,根据引理2,不等式(12)等价于不等式(10)。

其次,令 $\boldsymbol{e}(t)=\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)$,则 $(\boldsymbol{I}_2\otimes\boldsymbol{E})(\boldsymbol{I}_2\otimes(\boldsymbol{D}^{\alpha}\boldsymbol{e}(t)))=(\boldsymbol{I}_2\otimes(\boldsymbol{A}-\boldsymbol{LC}))(\boldsymbol{I}_2\otimes\boldsymbol{e}(t))$,$(\boldsymbol{I}_2\otimes\boldsymbol{E})(\boldsymbol{I}_2\otimes(D^{\alpha}\boldsymbol{x}(t)))=(\boldsymbol{I}_2\otimes(\boldsymbol{A}+\boldsymbol{BK}))(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))-(\boldsymbol{I}_2\otimes\boldsymbol{BK})(\boldsymbol{I}_2\otimes\boldsymbol{e}(t))$。将其代入式(9)中,则有

$$(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{x}(kT+i))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))+$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{x}(kT+i))+$$

$$(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{e}(kT+i))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\left(\sum_{j=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i))+$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i))^{\mathrm{T}}(\boldsymbol{I}_2\otimes E)^{\mathrm{T}}\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes D^{\alpha}\boldsymbol{e}(kT+i))=$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{E}D^{\alpha}\boldsymbol{x}(kT+i))^{\mathrm{T}}\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))+$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))^{\mathrm{T}}\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))(\boldsymbol{I}_2\otimes\boldsymbol{E}D^{\alpha}\boldsymbol{x}(kT+i))+$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{E}D^{\alpha}\boldsymbol{e}(kT+i))^{\mathrm{T}}\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i))+$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i))^{\mathrm{T}}\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))(\boldsymbol{I}_2\otimes\boldsymbol{E}D^{\alpha}\boldsymbol{e}(kT+i))=$$

$$(\boldsymbol{I}_2\otimes((\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{BK})\boldsymbol{x}(kT+i)-\boldsymbol{BKe}(kT+i)))^{\mathrm{T}}\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}$$

$$(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))+(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))^{\mathrm{T}}\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))$$

$$(\boldsymbol{I}_2\otimes((\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{BK})\boldsymbol{x}(kT+i)-\boldsymbol{BKe}(kT+i)))+(\boldsymbol{I}_2\otimes((\boldsymbol{A}+\Delta\boldsymbol{A}(t)-\boldsymbol{LC})\boldsymbol{e}(t)))^{\mathrm{T}}$$

$$\left(\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i))+(\boldsymbol{I}_2\otimes\boldsymbol{e}(kT+i))^{\mathrm{T}}$$

$$\sum_{i=1}^{2}(\boldsymbol{\Theta}_{i1}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij})+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))(\boldsymbol{I}_2\otimes((\boldsymbol{A}+\Delta\boldsymbol{A}(t)-\boldsymbol{LC})\boldsymbol{e}(kT+i)))\leqslant$$

$$-(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{Q}_1)(\boldsymbol{I}_2\otimes\boldsymbol{x}(kT+i))-(\boldsymbol{I}_2\otimes\boldsymbol{u}(kT+i))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{R})(\boldsymbol{I}_2\otimes\boldsymbol{u}(kT+i))。$$

该不等式等价于

$$\begin{bmatrix} I_2 \otimes x(kT+i) \\ I_2 \otimes e(kT+i) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^{\mathrm{T}} & \Omega_{22} \end{bmatrix} \begin{bmatrix} I_2 \otimes e(kT+i) \\ I_2 \otimes e(kT+i) \end{bmatrix} < 0, \tag{13}$$

其中

$$\Omega_{11} = \mathrm{sym}\left\{ \left(\sum_{i=1}^{2} \Theta_{i1} \otimes (E^{\mathrm{T}} P_0) + I_2 \otimes (Q^{\mathrm{T}} E_0^{\mathrm{T}}) \right) (I_2 \otimes (A + BK + \Delta A(t))) \right\} + I_2 \otimes (Q_1 + K^{\mathrm{T}} RK),$$

$$\Omega_{12} = \left(\sum_{i=1}^{2} \Theta_{i1} \otimes (E^{\mathrm{T}} P_0) + I_2 \otimes (Q^{\mathrm{T}} E_0^{\mathrm{T}}) \right) (I_2 \otimes (-BK)),$$

$$\Omega_{22} = \mathrm{sym}\left\{ (I_2 \otimes (A - LC + \Delta A(t)))^{\mathrm{T}} \left(\sum_{i=1}^{2} \Theta_{i1} \otimes (E^{\mathrm{T}} P_0) + I_2 \otimes (Q^{\mathrm{T}} E_0^{\mathrm{T}}) \right) \right\}。$$

根据不等式(13),有

$$\Phi = \mathrm{sym}\left\{ \sum_{i=1}^{2} \Theta_{i1} \otimes \left(E^{\mathrm{T}} P_0 \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \right) \right\} + \mathrm{sym}\left\{ I_2 \otimes \left(Q^{\mathrm{T}} E_0^{\mathrm{T}} \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \right) \right\} +$$

$$\mathrm{sym}\left\{ \sum_{i=1}^{2} \Theta_{i1} \otimes \left(E^{\mathrm{T}} P_0 \begin{bmatrix} \Delta A(t) & 0 \\ 0 & \Delta A(t) \end{bmatrix} \right) \right\} + \mathrm{sym}\left\{ I_2 \otimes \left(Q^{\mathrm{T}} E_0^{\mathrm{T}} \begin{bmatrix} \Delta A(t) & 0 \\ 0 & \Delta A(t) \end{bmatrix} \right) \right\} +$$

$$\begin{bmatrix} I_2 \otimes (Q_1 + K^{\mathrm{T}} RK) & 0 \\ 0 & 0 \end{bmatrix} < 0。 \tag{14}$$

其中,由引理 3 得

$$\mathrm{sym}\left\{ I_2 \otimes \left(Q^{\mathrm{T}} E_0^{\mathrm{T}} \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \right) \right\} = \mathrm{sym}\left\{ I_2 \otimes \left(Q^{\mathrm{T}} E_0^{\mathrm{T}} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right) \right\} +$$

$$\mathrm{sym}\left\{ I_2 \otimes (Q^{\mathrm{T}} E_0^{\mathrm{T}}) \left(I_2 \otimes \begin{bmatrix} BK & -BK \\ 0 & -LC \end{bmatrix} \right) \right\} \leqslant \mathrm{sym}\left\{ I_2 \otimes \left(Q^{\mathrm{T}} E_0^{\mathrm{T}} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \right) \right\} +$$

$$\left(I_2 \otimes (Q^{\mathrm{T}} E_0^{\mathrm{T}}) \right) (I_2 \otimes P_0)^{-1} \left(I_2 \otimes (Q^{\mathrm{T}} E_0^{\mathrm{T}}) \right)^{\mathrm{T}} + \left(I_2 \otimes \begin{bmatrix} BK & -BK \\ 0 & -LC \end{bmatrix} \right)^{\mathrm{T}} (I_2 \otimes P_0) \left(I_2 \otimes \begin{bmatrix} BK & -BK \\ 0 & -LC \end{bmatrix} \right)。$$

由 $F(t)F^{\mathrm{T}}(t) \leqslant I$,得$(I_2 \otimes F(t))(I_2 \otimes F(t))^{\mathrm{T}} = I_2 \otimes (F(t)F^{\mathrm{T}}(t)) \leqslant I$。又由于$\Theta_{11}\Theta_{11}^{\mathrm{T}} = I_2$,$\Theta_{21}\Theta_{21}^{\mathrm{T}} = I_2$,$\Delta A(t) = GF(t)H$,由引理 3,对于任意 $\varepsilon_1 > 0$,$\varepsilon_2 > 0$, 有

$$\mathrm{sym}\left\{ \sum_{i=1}^{2} \Theta_{i1} \otimes \begin{bmatrix} E^{\mathrm{T}} P_0 \Delta A(t) & 0 \\ 0 & E^{\mathrm{T}} P_0 \Delta A(t) \end{bmatrix} \right\} =$$

$$\mathrm{sym}\left\{ \left(I_2 \otimes \begin{bmatrix} E^{\mathrm{T}} P_0 G & 0 \\ 0 & E^{\mathrm{T}} P_0 G \end{bmatrix} \right) \left(I_2 \otimes \begin{bmatrix} F(t) & 0 \\ 0 & F(t) \end{bmatrix} \right) \left(\Theta_{i1} \otimes \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \right) \right\} \leqslant$$

$$\frac{1}{\varepsilon_i} \left(I_2 \otimes \begin{bmatrix} E^{\mathrm{T}} P_0 G & 0 \\ 0 & E^{\mathrm{T}} P_0 G \end{bmatrix} \right) \left(I_2 \otimes \begin{bmatrix} E^{\mathrm{T}} P_0 G & 0 \\ 0 & E^{\mathrm{T}} P_0 G \end{bmatrix} \right)^{\mathrm{T}} + \varepsilon_i \left(\Theta_{i1} \otimes \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \right)^{\mathrm{T}} \left(\Theta_{i1} \otimes \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \right) =$$

$$\frac{1}{\varepsilon_i} \left(I_2 \otimes \begin{bmatrix} E^{\mathrm{T}} P_0 G & 0 \\ 0 & E^{\mathrm{T}} P_0 G \end{bmatrix} \right) \left(I_2 \otimes \begin{bmatrix} E^{\mathrm{T}} P_0 G & 0 \\ 0 & E^{\mathrm{T}} P_0 G \end{bmatrix} \right)^{\mathrm{T}} + \varepsilon_i \left(I_2 \otimes \begin{bmatrix} H^{\mathrm{T}} H & 0 \\ 0 & H^{\mathrm{T}} H \end{bmatrix} \right)。$$

同理,对任意的 $\varepsilon_3 > 0$, 则得到

$$\mathrm{sym}\left\{ I_2 \otimes \left(Q^{\mathrm{T}} E_0^{\mathrm{T}} \begin{bmatrix} \Delta A(t) & 0 \\ 0 & \Delta A(t) \end{bmatrix} \right) \right\} \leqslant$$

$$\frac{1}{\varepsilon_3}\left(\boldsymbol{I}_2\otimes\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\begin{bmatrix}\boldsymbol{G}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{G}\end{bmatrix}\right)\left(\boldsymbol{I}_2\otimes\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\begin{bmatrix}\boldsymbol{G}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{G}\end{bmatrix}\right)^{\mathrm{T}}+\varepsilon_3\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\end{bmatrix}\right)。$$

代入式(14),得

$$\begin{aligned}\boldsymbol{\Phi}\leqslant\ &\mathrm{sym}\left\{\sum_{i=1}^{2}\boldsymbol{\Theta}_{i1}\otimes\begin{bmatrix}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0(\boldsymbol{A}+\boldsymbol{BK})&\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0(-\boldsymbol{BK})\\\boldsymbol{0}&\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0(\boldsymbol{A}-\boldsymbol{LC})\end{bmatrix}\right\}+\mathrm{sym}\left\{\boldsymbol{I}_2\otimes\left(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\begin{bmatrix}\boldsymbol{A}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{A}\end{bmatrix}\right)\right\}+\\&\sum_{i=1}^{3}\varepsilon_i\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\end{bmatrix}\right)+(\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))(\boldsymbol{I}_2\otimes\boldsymbol{P}_0)^{-1}(\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}))^{\mathrm{T}}+\\&\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{BK}&-\boldsymbol{BK}\\\boldsymbol{0}&-\boldsymbol{LC}\end{bmatrix}\right)^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{P}_0)\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{BK}&-\boldsymbol{BK}\\\boldsymbol{0}&-\boldsymbol{LC}\end{bmatrix}\right)+\\&\sum_{i=1}^{2}\frac{1}{\varepsilon_i}\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G}\end{bmatrix}\right)\left(\boldsymbol{I}_2\otimes\begin{bmatrix}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P}_0\boldsymbol{G}\end{bmatrix}\right)^{\mathrm{T}}+\\&\frac{1}{\varepsilon_3}\left(\boldsymbol{I}_2\otimes\left(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\begin{bmatrix}\boldsymbol{G}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{G}\end{bmatrix}\right)\right)\left(\boldsymbol{I}_2\otimes\left(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\begin{bmatrix}\boldsymbol{G}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{G}\end{bmatrix}\right)\right)^{\mathrm{T}}+\begin{bmatrix}\boldsymbol{I}_2\otimes(\boldsymbol{Q}_1+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{RK})&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{0}\end{bmatrix}。\end{aligned}\tag{15}$$

运用引理 25,则式(15)等价于式(11)。定理 1 得证。

为了解决定理 1 中的优化问题,借助 LMI 工具箱中的 mincx 求解器,求得观测器型控制器增益分别为 $\boldsymbol{K}=(\boldsymbol{P}_0^{-1})^{\mathrm{T}}\boldsymbol{X}_1,\boldsymbol{L}=\boldsymbol{P}_0^{-1}\boldsymbol{X}_2$。

3 鲁棒稳定性分析

引理 6[6] 对于定理 1 中的优化问题,在 kT 时刻的任意可行解在 $NT(N>k)$ 时刻仍是可行的。

引理 7[25] 假设$(\boldsymbol{E},\boldsymbol{A},\alpha)$是正则的,则系统是容许的当且仅当存在实对称正定矩阵 $\boldsymbol{Q}_{11},\boldsymbol{Q}_{21}$ 和斜对称矩阵 $\boldsymbol{Q}_{12},\boldsymbol{Q}_{22}$,以及矩阵 $\boldsymbol{Q}\in\mathbf{R}^{(n-r)\times n}$,使得

$$\mathrm{sym}\left\{\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij}\boldsymbol{A})\right)+\boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\boldsymbol{A})\right\}<0,$$

其中:$\theta=\pi\alpha/2,\mathrm{rank}(\boldsymbol{E})=r<n;\boldsymbol{E}_0\in\mathbf{R}^{n\times(n-r)}$ 是列满秩矩阵,且满足 $\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}_0=\boldsymbol{0};\boldsymbol{\Theta}_{11}=\begin{bmatrix}\sin\theta&-\cos\theta\\\cos\theta&\sin\theta\end{bmatrix}$,$\boldsymbol{\Theta}_{12}=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$,$\boldsymbol{\Theta}_{21}=\begin{bmatrix}\sin\theta&\cos\theta\\-\cos\theta&\sin\theta\end{bmatrix}$,$\boldsymbol{\Theta}_{22}=\begin{bmatrix}-\cos\theta&\sin\theta\\-\sin\theta&-\cos\theta\end{bmatrix}$。

本节主要探讨带有观测器的不确定分数阶广义系统的鲁棒稳定性。由定理 1 可确定状态反馈控制增益矩阵 $\boldsymbol{K}_k,k\in[0,+\infty)$,则闭环系统为:

$$(\boldsymbol{I}_2\otimes\bar{\boldsymbol{E}})(\boldsymbol{I}_2\otimes D^{\alpha}\bar{\boldsymbol{x}}(t))=(\boldsymbol{I}_2\otimes\bar{\boldsymbol{A}})(\boldsymbol{I}_2\otimes\bar{\boldsymbol{x}}(t)),$$

其中 $\bar{\boldsymbol{E}}=\begin{bmatrix}\boldsymbol{E}&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{E}\end{bmatrix},\bar{\boldsymbol{x}}(t)=\begin{bmatrix}\hat{\boldsymbol{x}}(t)\\\boldsymbol{e}(t)\end{bmatrix},\bar{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{A}+\boldsymbol{BK}_k&-\boldsymbol{BK}_k\\\boldsymbol{0}&\boldsymbol{A}-\boldsymbol{LC}\end{bmatrix}$。

根据引理 6,初始时刻的可行性条件确保了优化问题在所有时间内都具有可行性,从而为系统的稳定运行提供了理论依据。

定理 2 若优化问题(2)在初始时刻具有可行解,则由定理 1 得到的控制增益 $\boldsymbol{K}_k$ 使得基于观测器的闭环系统是可容许的。

证明 考虑带有误差项的 Lyapunov 函数:

$$\begin{aligned}&V(\boldsymbol{I}_2\otimes\boldsymbol{x}(t),\boldsymbol{I}_2\otimes\boldsymbol{e}(t))=\\&(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\boldsymbol{P}_k(\boldsymbol{I}_2\otimes\boldsymbol{x}(t))+(\boldsymbol{I}_2\otimes\boldsymbol{e}(t))^{\mathrm{T}}(\boldsymbol{I}_2\otimes\boldsymbol{E})^{\mathrm{T}}\boldsymbol{P}_k(\boldsymbol{I}_2\otimes\boldsymbol{e}(t)),\end{aligned}$$

其中 $\boldsymbol{P}_k = \left(\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij}) + \boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}^{\mathrm{T}}))\right)^{\mathrm{T}}$。根据不等式(9),可得

$$\mathrm{sym}\left\{\sum_{i=1}^{2}\sum_{j=1}^{2}(\boldsymbol{\Theta}_{ij}\otimes(\boldsymbol{E}^{\mathrm{T}}\boldsymbol{Q}_{ij}\bar{\boldsymbol{A}})) + \boldsymbol{I}_2\otimes(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{E}_0^{\mathrm{T}}\bar{\boldsymbol{A}})\right\} < 0,$$

其中 $\bar{\boldsymbol{A}} = \begin{bmatrix}\boldsymbol{A}+\boldsymbol{BK} & -\boldsymbol{BK}\\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{LC}\end{bmatrix}$。由引理 7 可知,闭环系统是可容许的。定理 2 得证。

4　仿真算例

考虑不确定分数阶系统(1),具体参数取为:

$$\boldsymbol{E} = \begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix},\boldsymbol{A} = \begin{bmatrix}-1&1&-1\\2&-2&-1\\4&1&-4\end{bmatrix},\boldsymbol{B} = \begin{bmatrix}1\\1\\1\end{bmatrix},\boldsymbol{C} = \begin{bmatrix}1\\0\\1\end{bmatrix}^{\mathrm{T}},\boldsymbol{G} = \begin{bmatrix}0.2&0.4&0.5\\0.3&0.2&0.1\\0.2&0.4&0.6\end{bmatrix},\boldsymbol{H} = \begin{bmatrix}0.2&0.3&0.1\\0.1&0.3&0.4\\0.1&0.2&0.4\end{bmatrix},$$

$$\boldsymbol{Q}_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\boldsymbol{R} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\boldsymbol{F}(t) = \mathrm{diag}\left(\sin(0.1\pi t),\sin(0.2\pi t),\cos(0.1\pi t)\right),\alpha = 0.6,$$

初始条件取为 $\boldsymbol{x}_0 = [0.2\quad 0\quad -0.1]^{\mathrm{T}}$。设 $\boldsymbol{E}_0 = [0\quad 0\quad 1]^{\mathrm{T}}$,利用 Matlab 中的 LMI 工具箱求解不等式(10)、(11),得到增益矩阵为 $\boldsymbol{K} = [0.107\,3\quad -0.354\,2\quad 2.657\,8]$,$\boldsymbol{L} = [-1.542\,3\quad 0.617\,2\quad -1.663\,2]^{\mathrm{T}}$。

在控制器的作用下,分数阶不确定系统的状态响应曲线和控制输入曲线如图 1、2 所示。在图 1 中,闭环系统的状态均收敛于零;图 2 显示系统的控制输入全局有界且收敛于零。因此,仿真结果表明本文设计的鲁棒预测控制算法的有效性,同时可以确保闭环系统的容许性。

5　结论

本文研究一类具有参数不确定性的分数阶广义系统的鲁棒预测控制问题,通过利用 Kronecker 积的性质对闭环系统进行了等价变换,推导得到观测器型鲁棒预测控制器存在的充分条件,并通过仿真结果验证了设计方案的有效性。当系统状态不完全可量测时,可以考虑设计基于输出反馈的鲁棒预测控制算法。

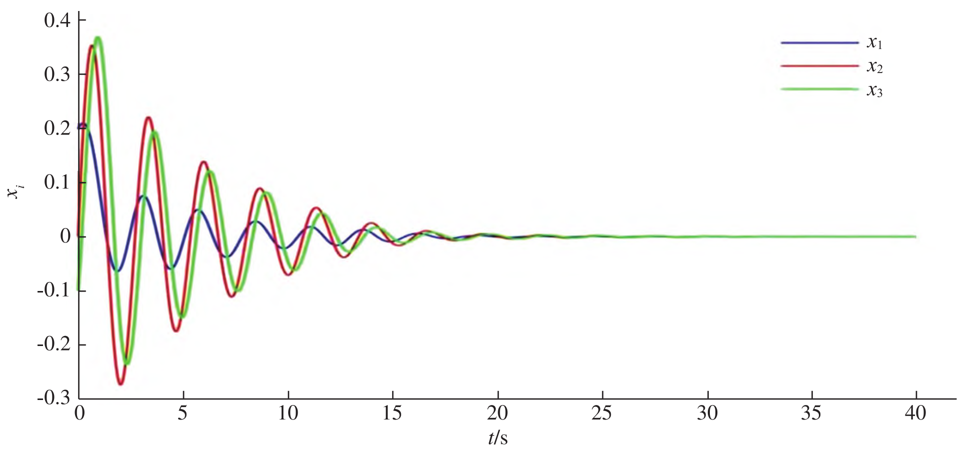

图 1　闭环系统的状态响应曲线

Fig.1 State response curves of the closed-loop system

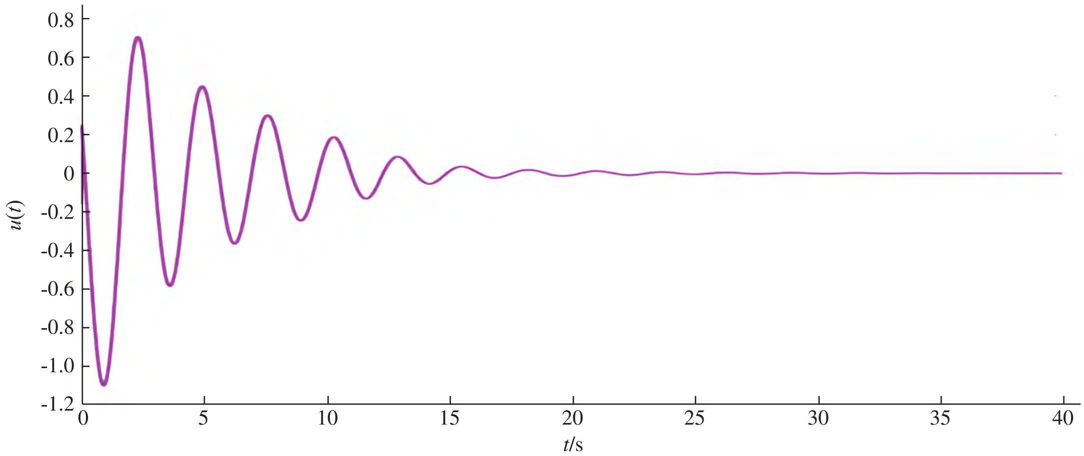

图2 控制输入变化曲线

Fig.2 The variation curve of the control input

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Robust Predictive Control for a Class of Uncertainty Fractional-order Singular Systems Based on Observer

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Abstract: In this paper, an observer-based robust predictive control strategy is investigated for a class of uncertain fractional-order singular systems. Firstly, based on the properties of fractional-order calculus, a Lyapunov function was constructed by incorporating appropriate error terms, and the associated optimization problem was addressed by using Lyapunov stability theorem. Secondly, by applying the linear matrix inequality and the cone complement linear algorithm, a sufficient condition for the existence of the robust predictive controller was established, while ensuring that the closed-loop system remains admissible. Finally, the effectiveness of the proposed control strategy was validated through simulation experiments.

Keywords: uncertain fractional-order singular systems; observer; robust predictive control; Lyapunov function